Lecture 13

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Related rates

If we are pumping air into a ballon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of radius

So in related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured.

The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Ex Air is being pumped into a spherical balloon so that it's volume increases at a rate of $100\ cm^3/s$. How fast is the radius of the balloon increasing when the radius is 50 cm ?

Soln

Given

The rate of increase of the volume is $100 \text{ cm}^3/\text{s}$

Unknown

The rate of increase of the radius when radius is 25 cm.

Set up let $V \equiv$ volume, $r \equiv$ radius

IMPORTANT RATES OF CHANGE ARE DERIVATIVES.

In this problem, volume and radius are both functions of time

The rate of increase of the volume with respect to time is $\frac{dV}{dt}$

and the rate of increase of the radius is $\frac{dr}{dt}$.

So Given $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$, and unknown $\frac{dr}{dt}$ when $r = 25$ cm

So final piece of information is how are V and r related, and

that is the volume of the sphere: $V = \frac{4}{3}\pi r^3$

Then finally we will differentiate both sides wrt t

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

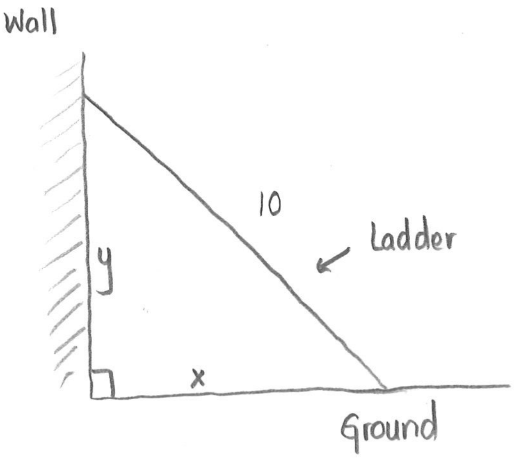
Then, $\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$

Then $r = 25$, $\frac{dV}{dt} = 100$ so we get

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2} \cdot 100 = \frac{1}{25\pi} \approx 0.0127 \text{ cm/s}$$

Ex 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6ft from the wall ?

First draw the diagram



Let $\underline{x}$ feet be the distance from the bottom of the ladder to the wall.

Let y feet be the distance from the top of the ladder to the ground.

- Note both x and y are functions of time.

Now, Given : $\frac{dx}{dt} = 1$ ft/s

unknown : $\frac{dy}{dt}$ when $x = 6$

The relation between x and y is given by the Pythagorean Thm

$$x^2 + y^2 = 10^2 \Rightarrow x^2 + y^2 = 100$$

Differentiating wrt t and using Chain rule,

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100)$$

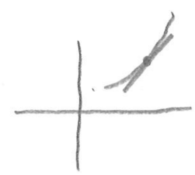
$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, by the Pythagorean Thm gives $y = 8$, so

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

2.8 Linear Approximations and differentials

- We have seen that a curve lies very close to it's tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like it's tangent line. This observation is the basis for a method of finding approximate values of functions.



- So we are going to use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. Equation of this tangent line is $y = f(a) + f'(a)(x-a)$ and the approximation

$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$ is called the linear approximation or tangent line approximation of f at a.

The Linear function whose graph is the tangent line, that is,

$L(x) = f(a) + f'(a)(x-a)$ is called the linearization of f at a.

Ex Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$

and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$?

Soln The derivative of $f(x) = \sqrt{x+3}$ is

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x+3}} .$$

Then $f(1) = 2$ and $f'(1) = \frac{1}{4}$

Then the linearization is $L(x) = f(1) + f'(x)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$

Therefore the linear approximation is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{when } x \text{ is near } 1$$

Then,

$$\sqrt{3.98} = \sqrt{0.98+3} = \frac{7}{4} + \frac{0.98}{4} = 1.995$$

$$\sqrt{4.05} = \sqrt{1.05+3} = \frac{7}{4} + \frac{1.05}{4} = 2.0125$$